Fourier Series

Lecture #12

- Midterm exam will be held on Wednesday 9 March 2011 at Auditorium Room from 06.45 – 08.00 pm.
- First order, second order, higher order, Laplace transform and power series.

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- ☐Introduction
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Introduction

Another type of series expansions for functions that can be helpful in computing solutions of DE is the Fourier series expansion.

Fourier series are in a certain sense more universal than the familiar Taylor series in calculus because many discontinuous periodic functions of practical interest can be developed in Fourier series but, of course, do not have Taylor series representations.

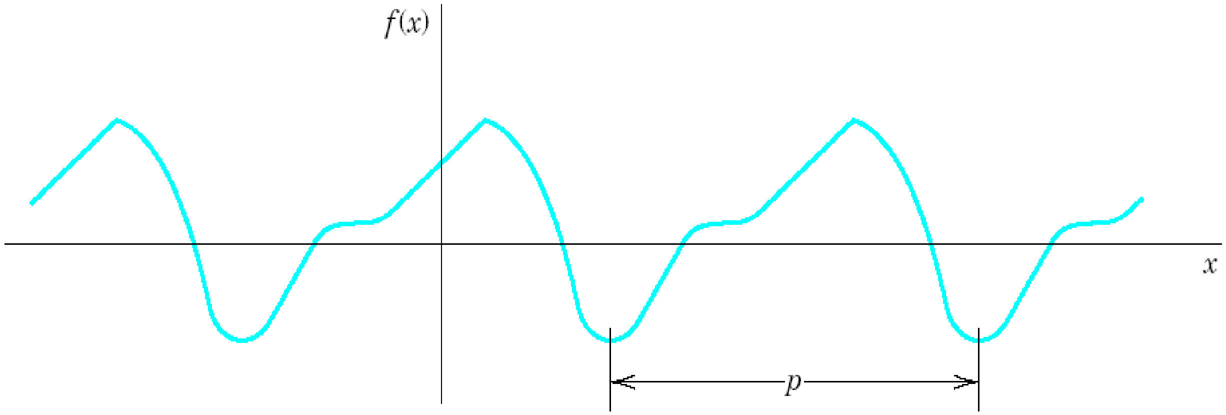
Introduction

Fourier series are infinite series designed to represent general periodic functions in term of simple ones: namely cosines and sinus. Thus, In general a function that has a Fourier series expansion will have to be periodic.

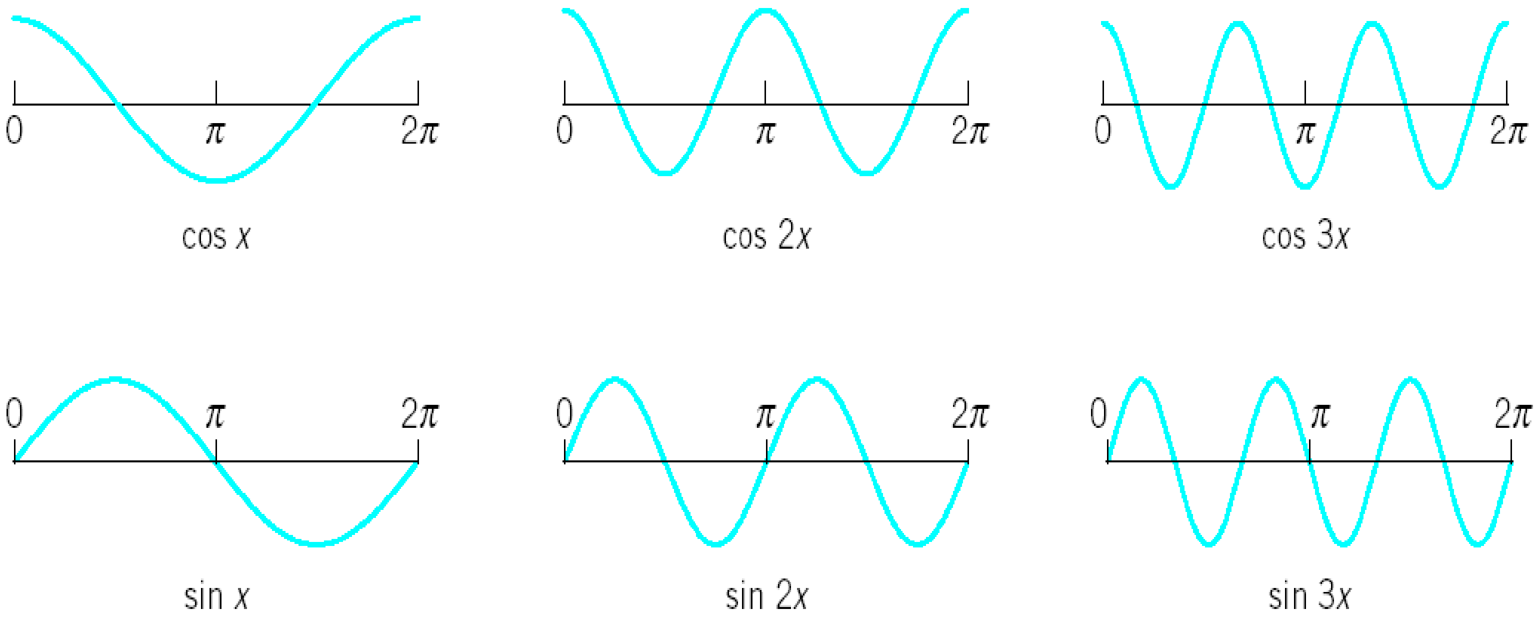
A function $f(x)$ defined for all real x is called **periodic** if there is some positive number p such that:

Periodic functions → $\boxed{f(x+p)=f(x)}$

Introduction



Periodic function



Cosine and sine functions are periodic

Fourier Series

-Definition-

Any periodic function *f(x)* of period 2π can be represented by a trigonometric series:

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

or

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We call this equation as the Fourier series of *f(x)*.

We will see the determination of Fourier coefficients of *f(x)*: a_0, a_n and b_n.

Fourier Series

To determine a_0, we integrate both sides from –π to π.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

$$= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-\pi}^{\pi}\cos nx\, dx}_{=0} + b_n \underbrace{\int_{-\pi}^{\pi}\sin nx\, dx}_{=0}\right]$$

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

Fourier Series

To determine a_n, we multiply by *cos(mx)* and integrate both from –π to π.

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx\,dx$$

$$= \underbrace{\int_{-\pi}^{\pi} a_0 \cos mx\,dx}_{=0} + \sum_{n=1}^{\infty}\left[\underbrace{\int_{-\pi}^{\pi} a_n(\cos nx \cos mx\,dx)}_{=a_n \pi} + \underbrace{\int_{-\pi}^{\pi} b_n(\sin nx \cos mx\,dx)}_{=0})\right]$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_n\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

Note: $m = n$

Fourier Series

To determine b_n, we multiply by *sin(mx)* and integrate both from –π to π.

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]\sin mx\,dx$$

$$= \underbrace{\int_{-\pi}^{\pi} a_0\sin mx\,dx}_{=0} + \sum_{n=1}^{\infty}\left[\underbrace{\int_{-\pi}^{\pi} a_n(\cos nx\sin mx\,dx)}_{=0} + \underbrace{\int_{-\pi}^{\pi} b_n(\sin nx\sin mx\,dx))}_{=b_n\,\pi}\right]$$

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_n\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Note: $m=n$

Fourier Series

For functions having period of 2π, the Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$ n=1,2,3,…..

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$ n=1,2,3,…..

Fourier Series: Summary

The functions considered have period of 2π, for simplicity of the formulas. In general, it can be applied to a function *f(x)* of period *p=2L*, then:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

n=1,2,3,…..

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

n=1,2,3,…..

Even and Odd functions

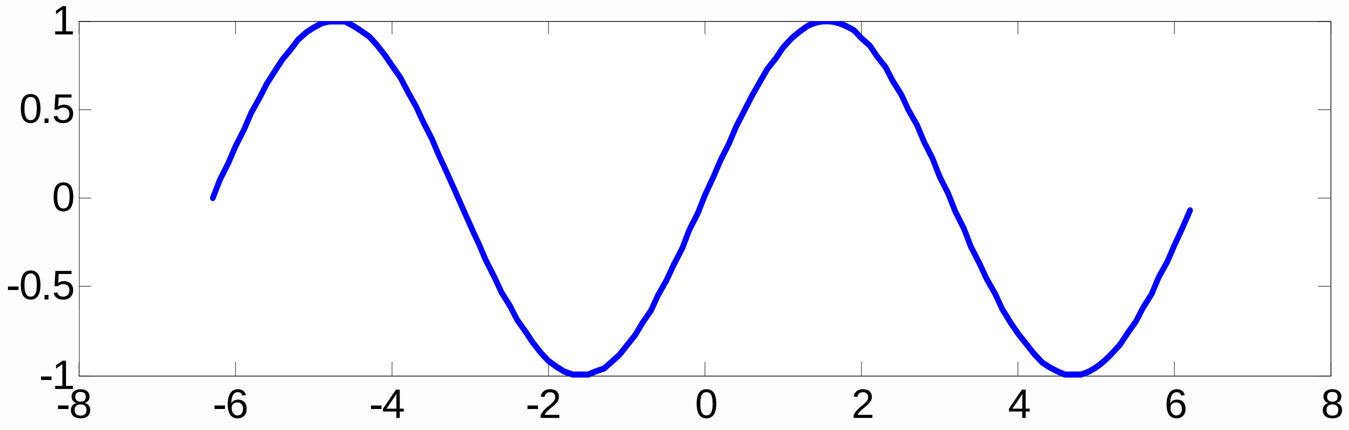
-Definition-

A function $g(x)$ is even if: $g(-x)=g(x)$

A function $h(x)$ is odd if: $h(-x)=-h(x)$

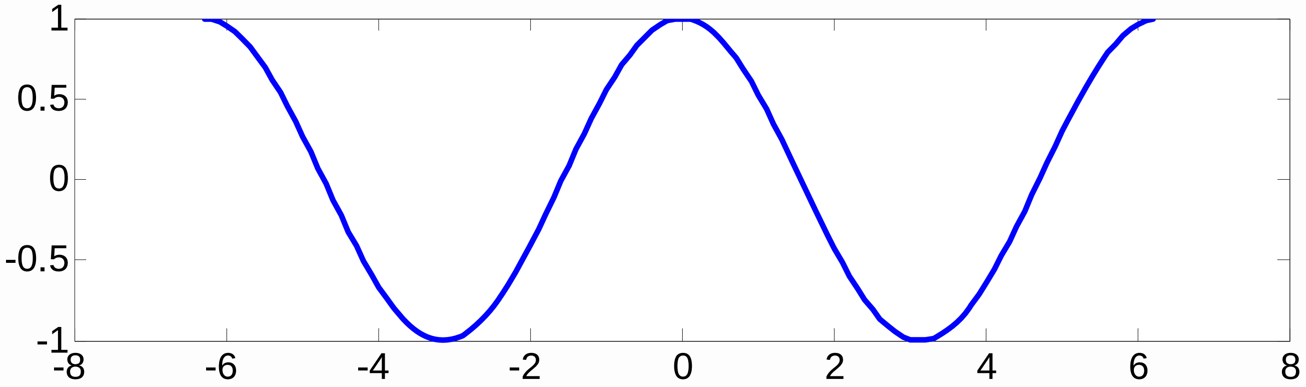
Example:

$\sin(nx)$ → odd function

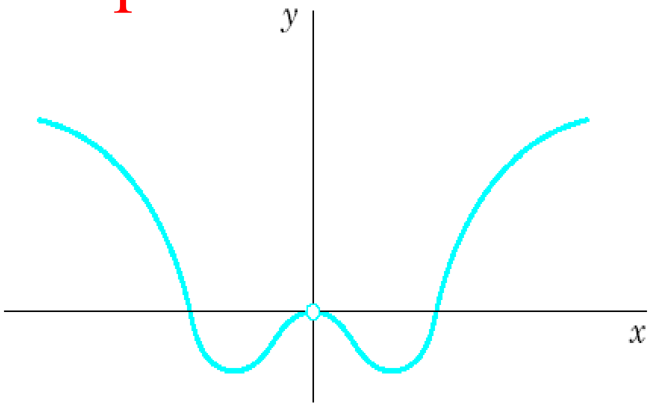


Even and Odd functions

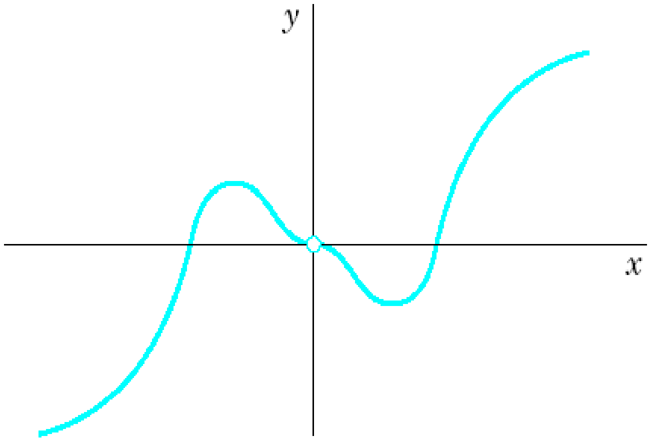
Example: $\cos(nx)$ → even function



Example:



Even function



Odd function

Even and Odd functions

-Some key facts-

1. If $g(x)$ is even function: $\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$

2. If $h(x)$ is odd function: $\int_{-L}^{L} h(x)dx = 0$

3. The product of an even and odd function is odd.

Proof:

Let: $q = g(x).h(x)$

$$q(-x) = g(-x).h(-x) = g(x).-h(x) = -g(x).h(x) = -q(x)$$

Fourier Series of Even and Odd functions

Let's see the Fourier series coefficients a_0, a_n and b_n:

If *f(x)* is even then *f(x).sin(..)* is odd. It implies $b_n=0$.

Similarly, *f(x)* is odd then *f(x).cos(..)* is odd. It implies $a_0=0$ and $a_n=0$.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\,dx$$

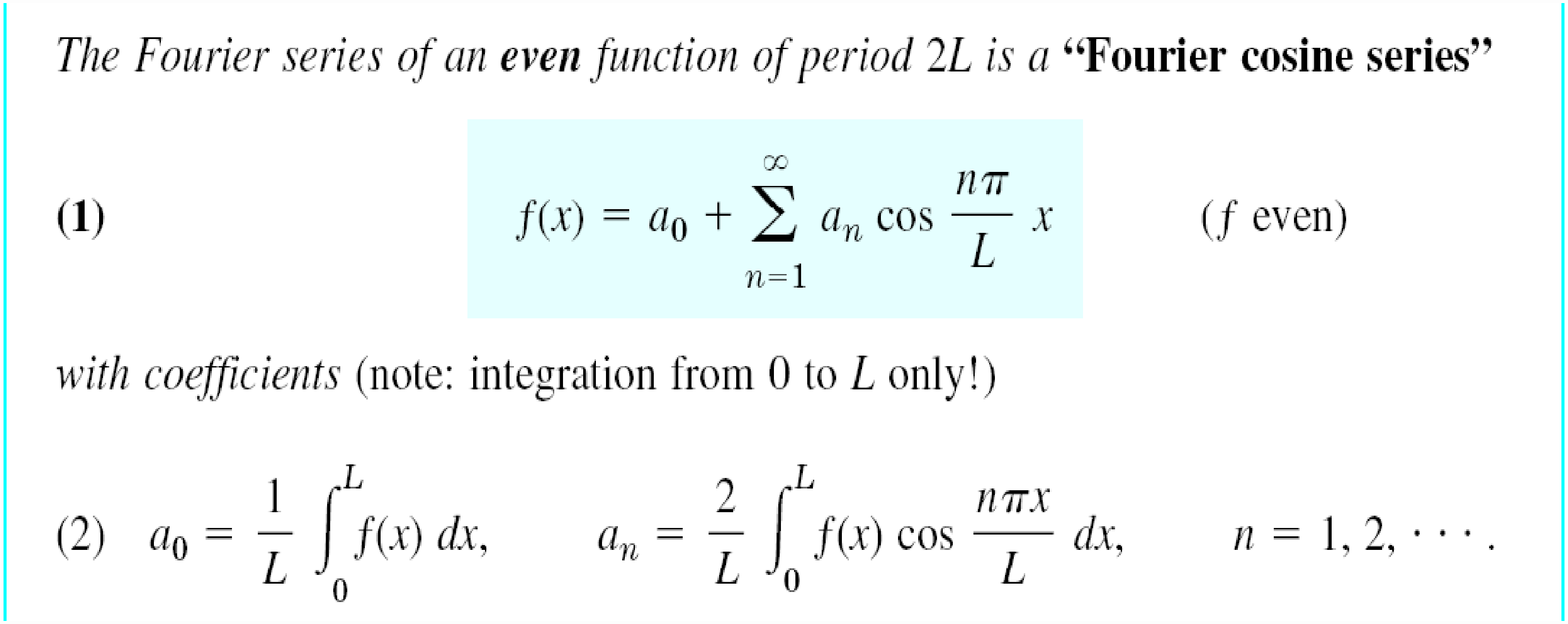
Note that:

sin(..) is odd.

cos(..) is even.

Fourier Cosine Series and Sine Series

Theorem



The Fourier series of an **even** *function of period 2L is a* **"Fourier cosine series"**

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$

with coefficients (note: integration from 0 to L only!)

(2) $$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots .$$

Fourier Cosine Series and Sine Series

Theorem

The Fourier series of an **odd** *function of period 2L is a* **"Fourier sine series"**

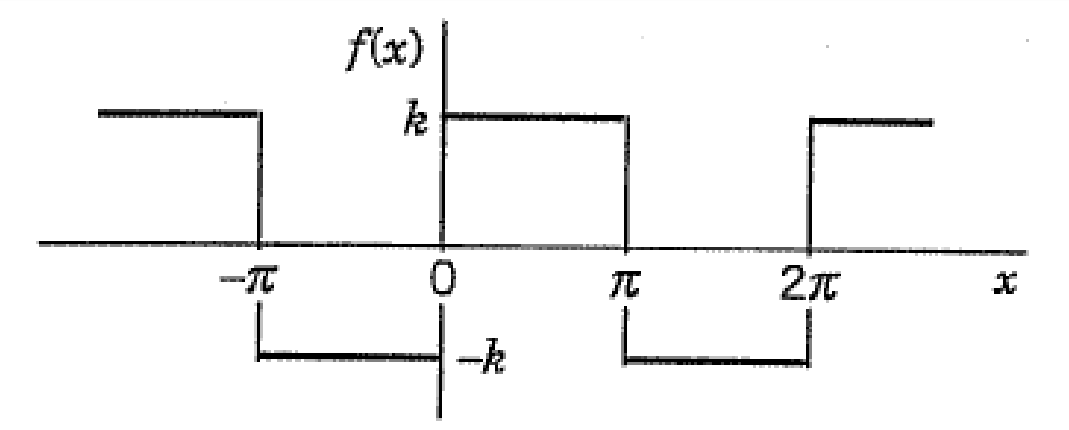
(3) $$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$

with coefficients

(4) $$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

Example 1:

Find the Fourier coefficients of the periodic function *f(x)* shown below.



$$f(x) = \begin{cases} -k & if \quad -\pi < x < 0 \\ k & if \quad 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$

Example 2:

Find the Fourier series of the periodic function $f(x)$ shown below.

$$f(x) = \begin{cases} 0 & if \quad -2 < x < -1 \\ k & if \quad -1 < x < 1 \\ 0 & if \quad 1 < x < 2 \end{cases}$$

Example 3:

Express $f(x)$ in a Fourier series:

$$f(x) = x + \pi \quad \text{for} \quad -\pi < x < \pi$$

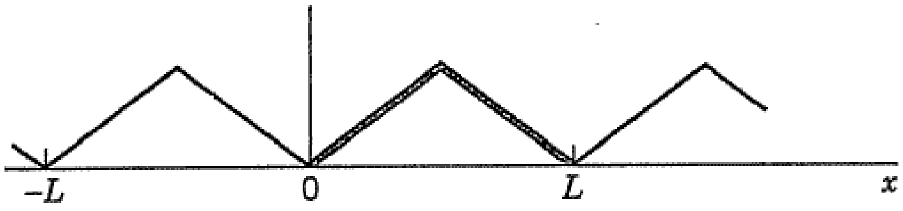
and given that $f(x+2\pi) = f(x)$

Example 4: Half-range Expansions

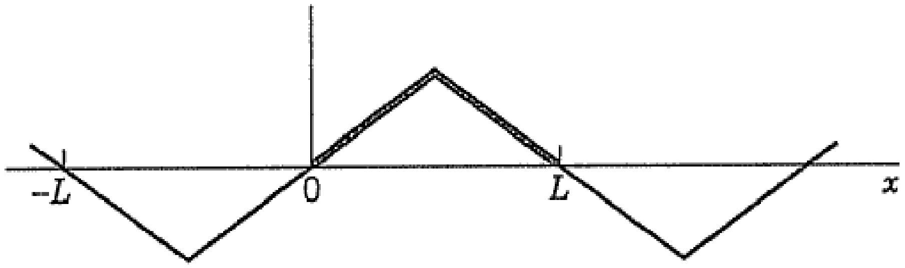
A half-range (half period) of a function is defined below.

Find its Fourier Sine series and Fourier Cosine series which represent respectively odd periodic extension and even periodic extension of the function.

$$f(x)=\begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0<x<\dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) & \text{if} \quad \dfrac{L}{2}<x<L \end{cases}$$



(a) Even extension



(b) Odd extension

Periodic extensions of $f(x)$ in Example 4

Both extensions have period 2L.

Example 4: solution

(a). Fourier cosine series coefficients

$$a_0 = \frac{k}{2}$$

$$a_n = \frac{4k}{n^2\pi^2}(2\cos\frac{n\pi}{2} - \cos n\pi - 1)$$

$$f(x) = \ldots\ldots.$$

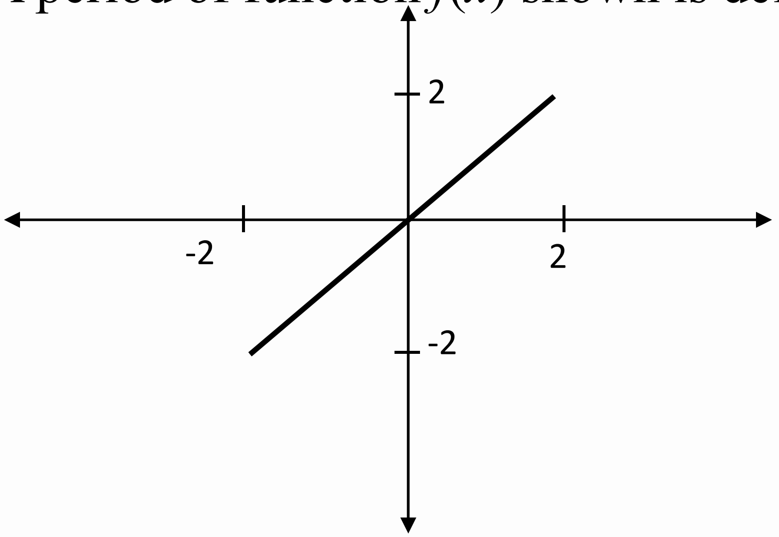
(b). Fourier sine series coefficients

$$b_n = \frac{8k}{n^2\pi^2}(\sin\frac{n\pi}{2})$$

$$f(x) = \ldots\ldots$$

Exercise:

1. A period of function $f(x)$ shown is defined over an interval $-2 \leq x \leq 2$.



(a) Define $f(x)$ based on the graph and determine whether $f(x)$ is even function, odd function or neither?

(b) Express $f(x)$ in a Fourier series.

2. Determine whether the given function is odd or even function and find its Fourier series.

$$f(x) = \frac{x^2}{2} \qquad \text{for } -\pi < x < \pi$$